Part V

Interaction with Underdense Plasmas - Nonlinear Wave Propagation and Wave-Breaking

5 Laser Propagation in Underdense Plasmas

Ionized gases: when is plasma response important?

- Simultaneous field ionization of many atoms produces a plasma with electron density n_e, temperature $T_e \sim 1 - 10$ eV.
- *Collective effects* important if

$$\omega_p \tau_L > 1$$

- Example: $\tau_L = 100$ fs, $n_e = 10^{17}$ cm^{-3} $\rightarrow \omega_p \tau_L = 1.8$
- Typical gas jets: $P \sim 1$bar; $n_e = 10^{18} - 10^{19}$ cm^{-3}
- Underdense: $\omega^2/\omega_p^2 = n_e/n_c \ll 1$; $n_c(1\mu) = 10^{21}$ cm^{-3}
- Exploit plasma effects for: (short-wavelength) radiation; high electric/magnetic fields; nonlinear refractive properties

Lorentz-Maxwell equations

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{p} = -e(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}), \quad (58)$$

$$\nabla \cdot \mathbf{E} = 4\pi e(n_0 - n_e), \quad (59)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad (60)$$

$$\nabla \times \mathbf{B} = -\frac{4\pi}{c} e n_e \mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad (61)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (62)$$

where $\mathbf{p} = \gamma m \mathbf{v}$ and $\gamma = (1 + p^2/m^2c^2)^{1/2}$.

Nonlinear plane-wave solutions
Akhiezer & Polovin, 1956

Look for solutions of the form $f(\omega t - \mathbf{k.r})$, or $f(\tau)$, where $\tau = t - \mathbf{i}\cdot\mathbf{r}/v_p$, and $v_p = \omega/k$ is the phase velocity of the wave. Temporal and spatial differential operators become:

$$
\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} \\
\nabla\cdot &= -\frac{\mathbf{i}}{v_p}\frac{\partial}{\partial \tau}\cdot \\
\nabla\times &= -\frac{\mathbf{i}}{v_p}\frac{\partial}{\partial \tau}\times\ ,
\end{aligned}
$$

where $\mathbf{i} = \mathbf{k}/\mid \mathbf{k} \mid$ is a unit vector in the direction of wave propagation.

Simplified Maxwell-fluid equations
– but still nonlinear!

$$\left(\frac{\mathbf{i}\cdot\mathbf{v}}{v_p}-1\right)\frac{d\mathbf{p}}{d\tau} = e\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right), \tag{63}$$

$$-\mathbf{i}\cdot\frac{d\mathbf{E}}{d\tau} = 4\pi e v_p(n_0-n_e), \tag{64}$$

$$\mathbf{B} = \frac{c}{v_p}\mathbf{i}\times\mathbf{E} \tag{65}$$

$$-\mathbf{i}\times\frac{d\mathbf{B}}{d\tau} = -\frac{4\pi}{c}e v_p n_e \mathbf{v}+\frac{v_p}{c}\frac{d\mathbf{E}}{d\tau}, \tag{66}$$

$$\mathbf{i}\cdot\frac{d\mathbf{B}}{d\tau} = 0\,. \tag{67}$$

NB: $\mathbf{i}\cdot\mathbf{B}=\mathbf{E}\cdot\mathbf{B}=0$ – B-field perpendicular to both the wave vector and E-field.

Electron density

Now take the dot product of Ampere Eq. (66) with the direction vector **i**:

$$\mathbf{i}\cdot\left(-\mathbf{i}\times\frac{d\mathbf{B}}{d\tau}\right)=\mathbf{i}\cdot\left(-\frac{4\pi}{c}ev_pn_e\mathbf{v}+\frac{v_p}{c}\frac{d\mathbf{E}}{d\tau}\right)$$

Eliminate **E** using Eq. (64) to give an equation for the density:

$$n_e=\frac{v_pn_0}{v_p-\mathbf{i}\cdot\mathbf{v}}\ . \tag{68}$$

Magnetic field

Similarly, the cross-product of the direction vector with Eq. (63) yields an explicit equation for **B**, namely:

$$\mathbf{B} = -\frac{c}{ev_p}\mathbf{i}\times\frac{d\mathbf{p}}{d\tau} \; . \tag{69}$$

Likewise, taking $\mathbf{i}\times(66)$ and making use of Eq. (65), we obtain an equation for $d\mathbf{B}/d\tau$:

$$\frac{d\mathbf{B}}{d\tau} = \frac{4\pi e n_e \beta_p}{\beta_p^2 - 1}\mathbf{i}\times\mathbf{v}, \tag{70}$$

where $\beta_p = v_p/c$.

Transverse wave equation

We can now eliminate **B** from the previous two equations by subtracting Eq. (70) from $d/d\tau$(69), leaving a *transverse wave equation*

$$\mathbf{i}\times\frac{d^2\mathbf{p}}{d\tau^2}+\frac{4\pi e^2 n_e\beta_p^2}{\beta_p^2-1}\mathbf{i}\times\mathbf{v}=0\,. \tag{71}$$

Longitudinal wave equation

Longitudinal component of the fluid motion (eliminate **B** from **i**·Eq. (63)):

$$\frac{d}{d\tau}\left[\left(\frac{\mathbf{i}\cdot\mathbf{v}}{v_p}-1\right)\mathbf{i}\cdot\frac{d\mathbf{p}}{d\tau}\right] = \frac{4\pi e^2 v_p n_0 \mathbf{i}\cdot\mathbf{v}}{v_p-\mathbf{i}\cdot\mathbf{v}} - \frac{1}{v_p}\frac{d}{d\tau}\left[\mathbf{v}\cdot\frac{d\mathbf{p}}{d\tau}-(\mathbf{i}\cdot\mathbf{v})(\mathbf{i}\cdot\frac{d\mathbf{p}}{d\tau})\right]. \tag{72}$$

Simplifications

To render equations (71) and (72) into something more tractable, we specify the wave vector $\mathbf{k}$ to be in the x-direction. Thus, we have $\mathbf{i} = \hat{x}$, $\mathbf{i} \cdot \mathbf{p} = p_x$ and $\mathbf{i} \times \mathbf{p} = (0, -p_z, p_y)$.
With these simplifications, and defining $\mathbf{u} = \mathbf{v}/c$, Eq. (68) becomes:

$$n_e = \frac{\beta_p n_0}{\beta_p - u_x}. \tag{73}$$

where $\beta_p = v_p/c, u_x = v_x/c$.
Typical feature of nonlinear plasma waves: density becomes very large in regions where the fluid velocity approaches the phase velocity.

Simplified wave equations – transverse

Taking the y and z components of Eq. (71) and making use of Eq. (73) and the usual definition $\omega_p^2 = 4\pi e^2 n_0/m$, gives us the coupled transverse wave equations:

$$\frac{d^2 p_z}{d\tau^2} + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1}\frac{\beta_p u_z}{\beta_p - u_x} = 0, \quad (74)$$

$$\frac{d^2 p_y}{d\tau^2} + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1}\frac{\beta_p u_y}{\beta_p - u_x} = 0, \quad (75)$$

where now p_y and p_z have been normalized to mc, so that $\mathbf{p} = \gamma\mathbf{u}$.

Simplified wave equations – longitudinal

The longitudinal wave equation follows from Eq. (72), which on applying the same choice of wave vector, simplifies to:

$$\frac{d}{d\tau}\left[(u_x - \beta_p)\frac{dp_x}{d\tau} + u_y\frac{dp_y}{d\tau} + u_z\frac{dp_z}{d\tau}\right] = \frac{\omega_p^2\beta_p^2 u_x}{\beta_p - u_x}\,. \qquad (76)$$

Closed set of equations (74–76) for nonlinear plasma waves of arbitrary amplitude and fixed phase velocity v_p.

Simplified wave equations – electric and magnetic fields

Once solved for **p**, the corresponding electric and magnetic fields (normalized here to $m\omega_p c/e$) can be obtained from:

$$\begin{aligned} E_x &= -\frac{1}{\beta_p}\frac{d}{d(\omega_p\tau)}\left(\beta_p p_x - (1+p^2)^{\frac{1}{2}}\right), \\ E_y &= -\frac{dp_y}{d(\omega_p\tau)}, \\ E_z &= -\frac{dp_z}{d(\omega_p\tau)}, \end{aligned} \tag{77}$$

$$\begin{aligned} B_x &= 0, \\ B_y &= \frac{1}{\beta_p}\frac{dp_z}{d(\omega_p\tau)}, \\ B_z &= -\frac{1}{\beta_p}\frac{dp_y}{d(\omega_p\tau)}. \end{aligned} \tag{78}$$

Potential: $\phi = \gamma - \beta_p p_x - 1$.

Low amplitude limit – linear dispersion relation

Transverse equation:

$$\frac{d^2 p_y}{d\tau^2} + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1} \frac{\beta_p u_y}{\beta_p - u_x} = 0$$

Linearize equation:

- $p_{x,y,z} \ll 1$
- $u_x \ll \beta_p$

$\longrightarrow$ neglect all terms $O(u^2)$ and higher

Geometry: EM wave propagates along the x-axis:

$$E_L = (0, E_y, 0),\ B_L = (0, 0, B_z),\ p_y = A_y.$$

Dispersion relation - transverse waves

Longitudinal and transverse wave components *decouple*, so that we recover the linear wave equation

$$\frac{d^2 u_y}{d\tau^2} + \frac{\omega_p^2 \beta_p^2 u_y}{\beta_p^2 - 1} = 0, \tag{79}$$

Solution:

$$u_y = u_o e^{-i\omega\tau},$$

provided that

$$-\omega^2 + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1} = 0.$$

Or, since $\beta_p = v_p/c = \omega/kc$, can rearrange to get:

$$\omega^2 = \omega_p^2 + c^2 k^2. \tag{80}$$

Dispersion relation - longitudinal (Langmuir) waves

Linearizing the longitudinal momentum equation (Eq. (76)) yields

$$-\beta_p \frac{d^2 p_x}{d\tau^2} - \omega_p^2 \beta_p u_x = 0,$$

or, since $\gamma \simeq 1$,

$$\frac{d^2 u_x}{d\tau^2} + \omega_p^2 u_x = 0. \tag{81}$$

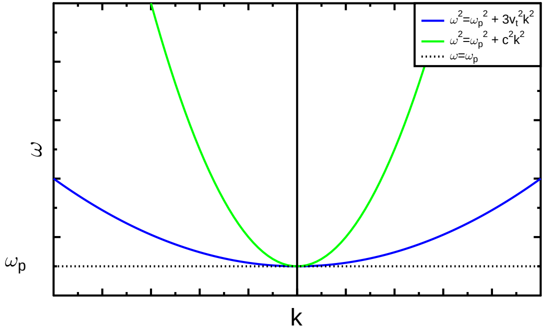
Solution: $u_x = u_{x0} e^{-i\omega\tau}$,with dispersion relation for longitudinal waves in the limit $T_e \to 0$.

$$\omega^2 = \omega_p^2.$$

Include finite temperature $T_e > 0$ – Bohm-Gross relation:

$$\omega^2 = \omega_p^2 + 3v_t^2 k^2. \tag{82}$$

Dispersion relations - graphical



Nonlinear plasma oscillations
Noble, 1984

$$\frac{d}{d\tau}\left[(u_x-\beta_p)\frac{dp_x}{d\tau}+u_y\frac{dp_y}{d\tau}+u_z\frac{dp_z}{d\tau}\right]=\frac{\omega_p^2\beta_p^2u_x}{\beta_p-u_x}\ .$$

Set $p_y = p_z = 0; u_x = u$ in Eq. (76), which simplifies to

$$\frac{d}{d\tau}\left[(u-\beta_p)\frac{dp}{d\tau}\right]=\frac{\omega_p^2\beta_p^2u}{\beta_p-u}.$$

Writing $p = \gamma u = u/\sqrt{1-u^2}$ and rearranging gives a 2nd order differential equation for the longitudinal velocity alone:

$$\frac{d^2}{d\tau^2}\left[\gamma(1-\beta_p u)\right]=\frac{\omega_p^2\beta_p^2u}{\beta_p-u}. \qquad (83)$$

Solution for longitudinal wave

This equation can be integrated once to give:

$$\frac{1}{2}Y^2 = \beta_p^2\omega_p^2(\gamma_m - \gamma),$$

where

$$Y = \frac{d}{d\tau}\left[\gamma(1-\beta_p u)\right],\ \gamma_m = (1-u_m^2)^{-1/2},$$

and $u_m = (v/c)_{max}$ is the maximum oscillation velocity of the wave. The waveform can thus be determined from the solution of:

$$\frac{d}{d\tau}\left[\gamma(1-\beta_p u)\right] = \pm\sqrt{2}\omega_p\beta_p(\gamma_m - \gamma)^{1/2}. \tag{84}$$

Once u is found, the density and electric field can immediately be determined using Eq. (73) and Eq. (77a) respectively:

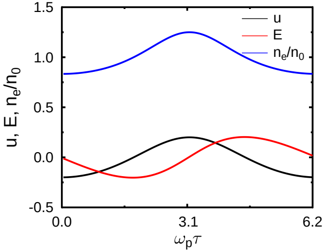
$$n_e(\tau) = \frac{\beta_p n_0}{\beta_p - u(\tau)}, \quad (85)$$

$$E(\tau) = \frac{Y}{\beta_p} = \pm\sqrt{2}(\gamma_m - \gamma(\tau))^{1/2}. \quad (86)$$

An exact analytical solution of Eqs. (84–2) can be obtained in the limiting case of $\beta_p = 1$, corresponding to a highly underdense plasma.

Numerical solutions – linear Langmuir wave

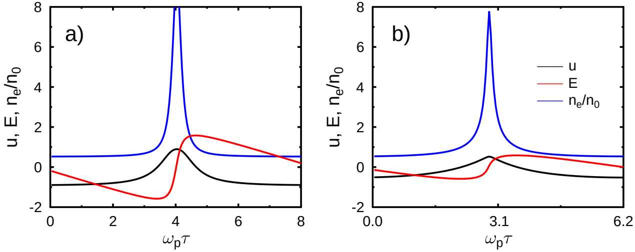
Integrate Eq. (84), Eq. (2) for $\beta_p = 1, u_m = 0.2$



NB: electric field and density 90^o out of phase

Numerical solutions – nonlinear Langmuir waves

Parameters: a) $\beta_p \simeq 1$ and $u_m = 0.9$; b) $\beta_p = 0.6, u_m = 0.55$



Typical features : i) sawtooth electric field; ii) spiked density; iii) *lengthening* of the oscillation period by factor γ.

Wavebreaking

- Definition: Wave breaks when the fluid velocity exceeds the phase velocity.
- If this happens, then some of the electron charge sheets may ‘cross’ each other, and the wave will lose its coherence.
- Analogy: surface water waves reaching shore.

Maximum electric field

Analyse electric field from 1D solution Eq. (2):

$$E(\tau) = \frac{Y}{\beta_p} = \pm\sqrt{2}(\gamma_m - \gamma(\tau))^{1/2}.$$

Extremum occurs for $\gamma = 1$, point in the oscillation when the electrons are momentarily stationary.
Thus at the wave-breaking point $\gamma_m = \gamma_p$, we have in physical units:

$$E_{\max} = \frac{mc\omega_p}{e}\sqrt{2}(\gamma_p - 1)^{1/2}. \qquad (87)$$

Maximum electric field - slow waves

For *non-relativistic* phase velocities $v_p \ll c$, we have

$$\gamma_p - 1 = (1 - \beta_p^2)^{-1/2} \simeq \beta_p^2/2,$$

so that

$$E_{\max} = \frac{m\omega_p v_p}{e}, \tag{88}$$

– cold wave-breaking limit (Dawson, 1962).

Maximum field amplitude - fast waves

For relativistic phase velocities, find

$$E_{\max} \sim \frac{m\omega_p c}{e}$$

Example

$$\begin{aligned} m_e &= 9.1 \times 10^{-28}\mathrm{g} \\ c &= 3 \times 10^{10}\mathrm{cms}^{-1} \\ \omega_p &= 5.6 \times 10^4 (n_e/cm^{-3})^{1/2} \\ e &= 4.8 \times 10^{-10}\mathrm{statcoulomb} \end{aligned} \tag{89}$$

Field:

$$E_p \sim 4 \times 10^8 \left(\frac{n_e}{10^{18}\ \mathrm{cm}^{-3}}\right)^{1/2} \mathrm{V\ m}^{-1}$$

Wavebreaking in warm plasmas

nonrelativistic - Coffey, 1971

Thermal effects act to reduce the maximum attainable wave amplitude, because:

1. plasma pressure resists the tendency for the density to explode
2. thermal electrons moving in the direction of the wave may be trapped at a lower wave amplitude than cold particles would be.

First analysed by Coffey (1971) using a so-called 'waterbag' model for the electron distribution function, giving:

$$\frac{eE_{\max}}{m\omega_p v_p} = \left(1 - \frac{\mu}{3} - \frac{8}{3}\mu^{1/4} + 2\mu^{1/2}\right)^{1/2}, \qquad (90)$$

where $\mu = 3k_B T_e/mv_p^2$.

Relativistic wavebreaking in warm plasmas

Katsouleas & Mori, 1988

Generalize waterbag model to include relativistic fluid momenta. Equivalent equation to Eq. (83) for the fluid velocity:

$$\frac{d^2}{d\tau^2}F(u) = \frac{\omega_p^2\beta_p^2 u}{\beta_p - u}, \tag{91}$$

where

$$F(u) = \frac{1-\beta_p u}{(1-u^2)^{1/2}}\left[1 + \mu\beta_p^2 \frac{1-u^2}{(\beta_p^2-u^2)^{1/2}}\right].$$

Relativistic wavebreaking in warm plasmas – electric field

Maximum electric field in the limit $\beta_p \simeq 1 \pm \sqrt{2}\omega_p \beta_p (\gamma_m - \gamma)^{1/2}$:

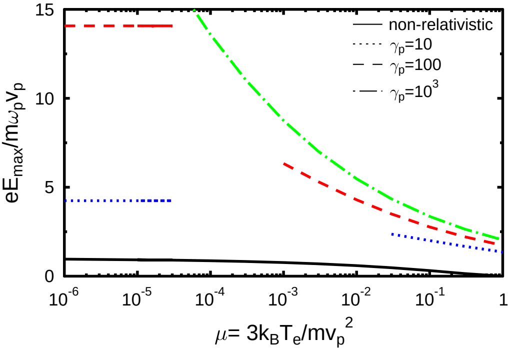
$$\frac{eE_{\max}}{m\omega_p c} = \mu^{-1/4}(\ln 2\gamma_p^{1/2}\mu^{1/4})^{1/2}, \tag{92}$$

valid for

$$\gamma_p \gg \frac{1}{2\mu^{1/2}} \log 2\mu^{1/4}\gamma_p^{1/2}.$$

Maximum electric field in warm plasmas

Wavebreaking amplitude of longitudinal plasma oscillations for different phase velocities.
Numerical solutions join up with the cold wavebreaking limits (shown for $\gamma_p = 10$ and $\gamma_p = 100$).



Summary of Lecture 5

- Plasma can support large amplitude waves
- Longitudinal & transverse components generally *coupled*
- Longitudinal waves: $\longrightarrow$ spiky electron density; sawtooth E-field
- Max electric field determined by *wave breaking limit*:

$$E_{\mathrm{max}} \sim \frac{m\omega_p c}{e}$$