Part VI

Interaction with Underdense Plasmas - Wakefield Excitation

Wakefield excitation

EM wave propagation is described by the transverse wave equation (74):

$$\frac{d^2 p_y}{d\tau^2} + \frac{\omega_p^2 \beta_p^2}{\beta_p^2 - 1} \frac{\beta_p u_y}{\beta_p - u_x} = 0$$

Plasma wave governed by longitudinal motion (76):

$$\frac{d}{d\tau}\left[(u_x - \beta_p)\frac{dp_x}{d\tau} + u_y \frac{dp_y}{d\tau}\right] = \frac{\omega_p^2 \beta_p^2 u_x}{\beta_p - u_x}.$$

How we can drive plasma waves with laser pulses?

Wakefield excitation - small pump strengths

Recall that Eqs. (74) and (76) are coupled through the nonlinear terms. Retain a nonlinear *pump* term in Eq. (76) to give:

$$\frac{d^2u_x}{d\tau^2} + \omega_p^2 u_x \quad \simeq \quad \frac{1}{\beta_p}\frac{d^2}{d\tau^2}\left(\frac{u_y^2}{2}\right), \tag{93}$$

where we have set $p_y = \gamma u_y \simeq u_y$ if $u_{x,y} \ll 1$.

$\Rightarrow$ Driven oscillator: pump strength $\propto$ laser intensity!

$$n_e = \frac{\beta_p n_0}{\beta_p - u_x}$$

===> $$n = n_e - n_0 = \frac{\beta_p n_0 - n_0(\beta_p - u_x)}{\beta_p - u_x} = \frac{n_0 u_x}{\beta_p - u_x} \approx \frac{n_0 u_x}{\beta_p}$$

$$\frac{d^2 n}{d\tau^2} + \omega_p^2 n = \frac{n_0}{\beta_p^2} \frac{d^2}{d\tau^2} \frac{u_y^2}{2}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = -\frac{1}{c\beta_p} \frac{\partial}{\partial \tau}$$

$$\frac{\partial^2 n}{\partial t^2} + \omega_p^2 n = \frac{n_0}{2} \frac{\partial^2}{\partial x^2} \frac{v_y^2}{c}$$

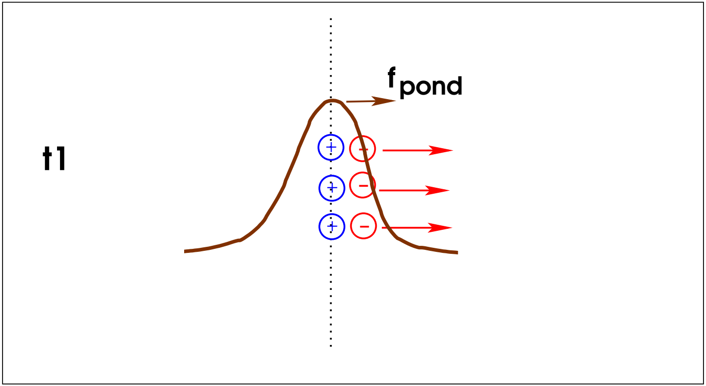
Wakefield excitation: physical picture

- Plasma wave is driven by longitudinal *ponderomotive force*, which pushes electrons away from regions of high intensity.
- When pulse enters fresh plasma, electrons initially pushed forward.
- After pulse maximum, ponderomotive force *reverses* sign

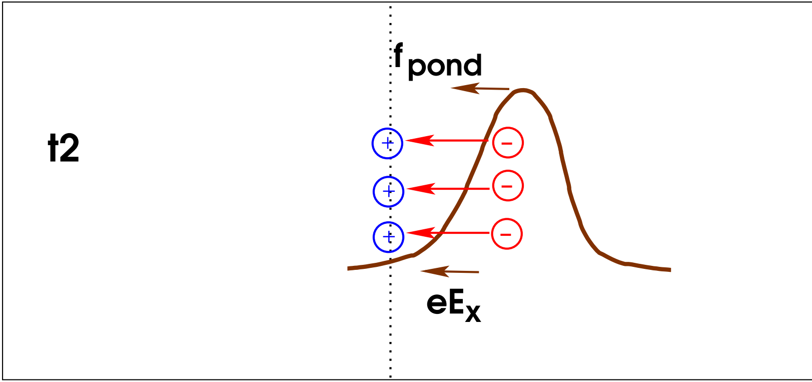
$\Rightarrow$ electrons receive another kick in the opposite direction

– see Fig. 4

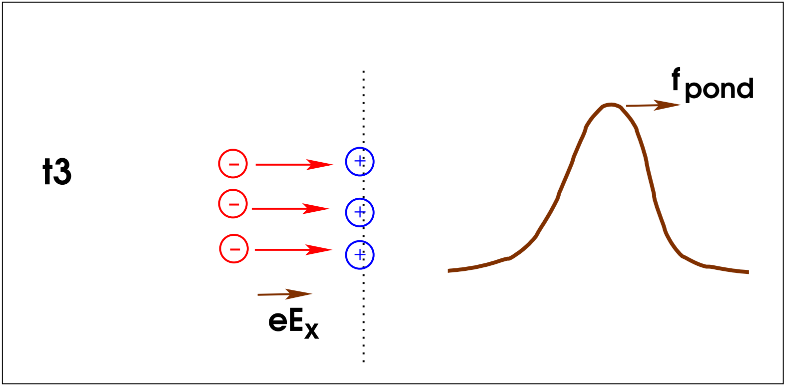
Ponderomotive kick: $t = 0$



Ponderomotive kick: $t = \pi/\omega_p$



Ponderomotive kick: $t = 2\pi/\omega_p$



Resonance condition

The amplitude of the longitudinal oscillation will be enhanced if the pulse length is roughly matched to the plasma period:

$$\tau_L \simeq \omega_p^{-1}.$$

Example

What plasma density do we need to match a 100 fs pulse?

$$\omega_p \simeq 5 \times 10^4 n_e^{1/2} \;\; s^{-1}$$

Matching condition:

$$n_e \simeq 4 \times 10^{14} \tau_{\rm ps}^{-2} \text{ cm}^{-3}$$

For 100 fs, need $n_e = 4 \times 10^{16}$ cm^{-3}.

Quasistatic approximation: speedboat model of wakefields

- What wake amplitude can we expect?
- Analyse using coordinate transformation to a frame moving with the *group velocity* of the laser pulse $v_g \simeq c$.
- Choose variables (ξ, τ) such that:

$$\xi = x - ct, \tau = t.$$

Quasi-static variables

- Partial derivatives then become:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}; \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - c\frac{\partial}{\partial \xi} \simeq -c\frac{\partial}{\partial \xi} \tag{94}$$

- The time τ is considered to be slowly varying during the transit time of the pulse – typically the Rayleigh diffraction time

$$t_R = R_L/c = \frac{k_0 \sigma_L^2}{c} \gg 2\pi/\omega_0.$$

  Can then set $\partial/\partial\tau = 0$ in this 'co-moving' frame.
- NB: Eulerian, (not Lorentz) transformation.

Driven oscillator within QSA

Electron density (from 93 with $n \simeq n_0 u_x/\beta_p$), or directly from Maxwell's equations:

$$\left(\frac{\partial^2}{\partial\xi^2} + k_p^2\right) n = \frac{n_0}{2}\frac{\partial^2}{\partial\xi^2} a^2. \tag{95}$$

Use Poisson's equation to get electric field and potential:

$$\frac{\partial^2\phi}{\partial\xi^2} = -\frac{\partial E}{\partial\xi} = 4\pi e n,$$

Wake electric field and potential

E normalized to $m\omega_p c/e$; *Phi* to mc/e:

$$\left(\frac{\partial^2}{\partial\xi^2} + k_p^2\right) E = k_p^2 \frac{\partial}{\partial\xi}\Phi_L, \quad (96)$$

$$\left(\frac{\partial^2}{\partial\xi^2} + k_p^2\right) \phi = -k_p^2 \Phi_L, \quad (97)$$

where $\Phi_L = -\frac{1}{2} < a^2 >$ is the normalized ponderomotive potential of the laser pulse, averaged over the laser period $2\pi/\omega_0$.

Wakefield: solution

Eq. (97) is a driven Helmholtz equation – solve with Green function methods.
Formal solution:

$$\phi(\xi) = -\frac{k_p}{4} \int_{\xi}^{\infty} d\xi' \mid a(\xi') \mid^2 \sin[k_p(\xi - \xi')]. \qquad (98)$$

Wakefield: solution for $\sin^2$ pulse

Consider '$\sin^2$'-pulse:

$$a^2(\xi) = \begin{cases} a_0^2 \sin^2(\frac{\pi\xi}{\xi_L}), & 0 \le \xi \le \xi_L \\ 0, & \xi < 0, \xi > \xi_L \end{cases}$$

Behind the pulse ($\xi < 0$), have:

$$\phi(\xi) = \frac{2\pi^2 \Phi_L}{(4\pi^2 - k_p^2 \xi_L^2)} \left[\cos k_p(\xi - \xi_L) - \cos k_p \xi\right]. \qquad (99)$$

Solution behind pulse: Wake E-field

The longitudinal electric field left behind by the pulse is then simply:

$$
\begin{aligned}
E_z &= -\frac{\partial \phi}{\partial \xi} \\
&= \frac{2\pi^2 \Phi_L k_p}{(4\pi^2 - k_p^2 \xi_L^2)} \left[\sin k_p(\xi - \xi_L) - \sin k_p \xi \right]. \qquad (100)
\end{aligned}
$$

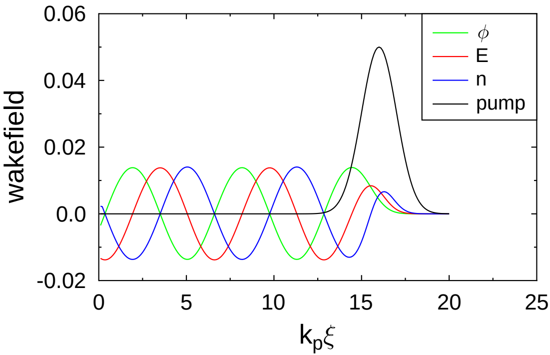
Wakefield – resonance condition II

Eq. (100) has a maximum (resonance) for $k_p\xi_L = 2\pi$, or equivalently, for pulse lengths $\xi_L = \lambda_p$.
Using l'Hospital's rule:

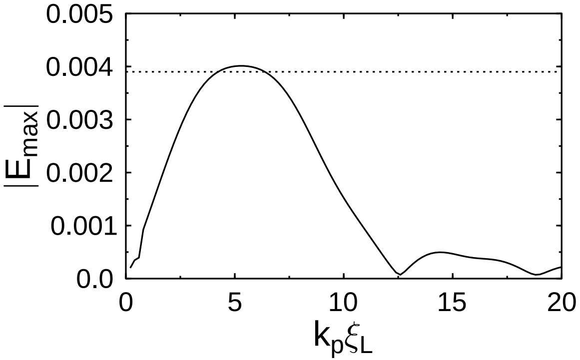
$$E_z^{\max}(\xi) = \frac{\pi^2 \Phi_L}{\lambda_p} \cos k_p\xi, \tag{101}$$

– scales with the laser intensity, or a_0^2.

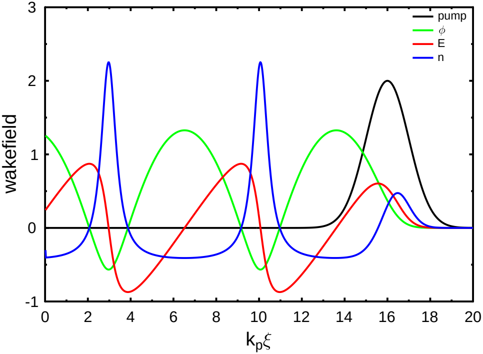
Numerical solution: small laser amplitude



Numerical solution: resonance condition (small amplitudes)



Numerical solution: large laser amplitude



Wake amplitude scaling in nonlinear regime

Murusidze & Berzhiani, 1990

Analytical solution possible for a *square* pump in the limit $\beta_g \to 1 \;\Rightarrow$ Scaling of the wake-variable maxima:

$$
\begin{aligned}
\phi_{\max} &\sim \gamma_\perp^2 - 1 \\
E_{\max} &\sim \frac{\gamma_\perp^2 - 1}{\gamma_\perp} \\
p_{\max} \sim (\gamma u)_{\max} &= \frac{\gamma_\perp^4 - 1}{2\gamma_\perp^2}
\end{aligned}
\qquad (102)
$$

where $\gamma_\perp = (1 + a^2)^{1/2}$ as on p. **??**.

Electron acceleration by wakefields

- Conventional synchrotrons and LINACS operate with field gradients limited to around 100 MVm^{-1}.
- Plasma is already ionized; can theoretically sustain a field 10^4 times larger, given by:

$$\begin{aligned} E_p &= \frac{m_e c \omega_p}{e}\varepsilon \\ &\simeq n_{18}^{1/2}\varepsilon \ \mathrm{GV\ cm}^{-1}, \end{aligned} \tag{103}$$

where n_{18} is the electron density in units of 10^{18} cm^{-3}.

Laser-electron accelerator
Tajima & Dawson, 1979

Laser-driven wakefields must propagate with velocities approaching the speed of light ($v_p = v_g < c$).
Plasma wave has a phase velocity:

$$v_p = c\left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^{\frac{1}{2}} \simeq c\left(1 - \frac{1}{2\gamma_p^2}\right), \tag{104}$$

where $\gamma_p = \omega_0^2/\omega_p^2$.

Acceleration length

A relativistic electron ($v \simeq c$) trapped in such a wave will be accelerated over at most *half a wavelength* in the wave-frame, after which it starts to be *decelerated*.
Effective acceleration length:

$$
\begin{aligned}
L_a = \frac{\lambda_p c}{2(c - v_p)} &\simeq \lambda_p \gamma_p^2 \\
&= \frac{\omega^2}{\omega_p^2} \lambda_p \\
&\simeq 3.2\, n_{18}^{-3/2} \lambda_{\mu\mathrm{m}}^{-2} \ \text{cm}. \qquad (105)
\end{aligned}
$$

Maximum energy gain

Combine Eq. (103) and Eq. (105) to obtain the maximum energy gain:

$$\begin{aligned}\Delta U &= eE_p.L_a \\ &= e\left(\frac{m\omega_p c}{e}\right)\varepsilon\frac{\omega^2}{\omega_p^2}\frac{2\pi c}{\omega_p} \\ &= 2\pi\left(\frac{\omega}{\omega_p}\right)^2\varepsilon\, mc^2 \\ &\simeq 3.2\, n_{18}^{-1}\lambda_{\mu\mathrm{m}}^{-2}\ \text{GeV}. \end{aligned} \tag{106}$$

Limiting factors

In principle, TW laser is capable of accelerating an electron to 5 GeV in a distance of 5 cm through a plasma with density 10^{18} cm^{-3}.
Spoiling factors:

- Diffraction: typically have $Z_R \ll L_a$, so some means of *guiding* the laser beam over the dephasing length is essential
- Propagation instabilities – beam break-up: modulation; hosing; Raman