221A Lecture Notes
Notes on Tensor Product

1 What is "Tensor"?

After discussing the tensor product in the class, I received many questions what it means. I've also talked to Daniel, and he felt this is a subject he had learned on the way here and there, never in a course or a book. I myself don't remember where and when I learned about it. It may be worth solving this problem once and for all.

Apparently, Tensor is a manufacturer of fancy floor lamps. See `http://growinglifestyle.shop.com/cc.class/cc?pcd=7878065&ccsyn=13549`.

For us, the word "tensor" refers to objects that have multiple indices. In comparison, a "scalar" does not have an index, and a "vector" one index. It appears in many different contexts, but this point is always the same.

2 Direct Sum

Before getting into the subject of tensor product, let me first discuss "direct sum." This is a way of getting a new big vector space from two (or more) smaller vector spaces in the simplest way one can imagine: you just line them up.

2.1 Space

You start with two vector spaces, V that is n-dimensional, and W that is m-dimensional. The tensor product of these two vector spaces is $n+m$-dimensional. Here is how it works.

Let $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ be the basis system of V, and similarly $\{\vec{f}_1, \vec{f}_2, \cdots, \vec{f}_m\}$ that of W. We now define $n+m$ basis vectors $\{\vec{e}_1, \cdots, \vec{e}_n, \vec{f}_1, \cdots, \vec{f}_m\}$. The vector space spanned by these basis vectors is $V \oplus W$, and is called the direct sum of V and W.

2.2 Vectors

Any vector $\vec{v} \in V$ can be written as a linear combination of the basis vectors, $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$. We normally express this fact in the form of a column vector,

$$\vec{v} = \left.\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_V \right\} n. \tag{1}$$

Here, it is emphasized that this is an element of the vector space V, and the column vector has n components. In particular, the basis vectors themselves can be written as,

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_V, \qquad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}_V, \qquad \cdots, \qquad \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_V. \tag{2}$$

If the basis system is orthonormal, *i.e.*, $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$, then $v_i = \vec{e}_i \cdot \vec{v}$. Similarly for any vector w, we can write

$$\vec{w} = \left.\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}_W \right\} m. \tag{3}$$

The basis vectors are

$$\vec{f}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_W, \qquad \vec{f}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}_W, \qquad \cdots, \qquad \vec{f}_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_W. \tag{4}$$

The vectors are naturally elements of the direct sum, just by filling zeros to the unused entries,

$$\vec{v} = \left.\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}\right\} n+m, \qquad \vec{w} = \left.\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline w_1 \\ \vdots \\ w_m \end{pmatrix}\right\} n+m. \tag{5}$$

One can also define a direct sum of two non-zero vectors,

$$\vec{v} \oplus \vec{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ \hline w_1 \\ \vdots \\ w_m \end{pmatrix} = \left.\begin{pmatrix} \vec{v} \\ \hline \vec{w} \end{pmatrix}\right\} n+m, \tag{6}$$

where the last expression is used to save space and it is understood that $\vec{v}$ and $\vec{w}$ are each column vector.

2.3 Matrices

A matrix is mathematically a linear map from a vector space to another vector space. Here, we specialize to the maps from a vector space to the same one because of our interest in applications to quantum mechanics, $A : V \to V$, *e.g.*,

$$\vec{v} \mapsto A\vec{v}, \tag{7}$$

or

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_V \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_V \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_V . \tag{8}$$

Similarly, $B : W \to W$, and $B : \vec{w} \mapsto B\vec{w}$.

On the direct sum space, the same matrices can still act on the vectors, so that $\vec{v} \mapsto A\vec{v}$, and $\vec{w} \mapsto B\vec{w}$. This matrix is written as $A \oplus B$. This can be achieved by lining up all matrix elements in a *block-diagonal* form,

$$A \oplus B = \left(\begin{array}{c|c} A & 0_{n\times m} \\ \hline 0_{m\times n} & B \end{array} \right). \tag{9}$$

For instance, if $n = 2$ and $m = 3$, and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_V, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}_W, \tag{10}$$

where it is emphasized that A and B act on different spaces. Their direct sum is

$$A \oplus B = \left(\begin{array}{cc|ccc} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{array} \right). \tag{11}$$

The reason why the block-diagonal form is appropriate is clear once you act it on $\vec{v} \oplus \vec{w}$,

$$(A \oplus B)(\vec{v} \oplus \vec{w}) = \left(\begin{array}{c|c} A & 0_{n\times m} \\ \hline 0_{m\times n} & B \end{array} \right) \left(\begin{array}{c} \vec{v} \\ \hline \vec{w} \end{array} \right) = \left(\begin{array}{c} A\vec{v} \\ \hline B\vec{w} \end{array} \right) = (A\vec{v}) \oplus (B\vec{w}). \tag{12}$$

If you have two matrices, their multiplications are done on each vector space separately,

$$(A_1 \oplus B_1)(A_2 \oplus B_2) = (A_1 A_2) \oplus (B_1 B_2). \tag{13}$$

Note that not every matrix on $V \oplus W$ can be written as a direct sum of a matrix on V and another on W. There are $(n+m)^2$ independent matrices on $V \oplus W$, while there are only n^2 and m^2 matrices on V and W, respectively. Remaining $(n+m)^2 - n^2 - m^2 = 2nm$ matrices cannot be written as a direct sum.

Other useful formulae are

$$\det(A \oplus B) = (\det A)(\det B), \tag{14}$$

$$\mathrm{Tr}(A \oplus B) = (\mathrm{Tr} A) + (\mathrm{Tr} B). \tag{15}$$

3 Tensor Product

The word "tensor product" refers to another way of constructing a big vector space out of two (or more) smaller vector spaces. You can see that the spirit of the word "tensor" is there. It is also called Kronecker product or direct product.

3.1 Space

You start with two vector spaces, V that is n-dimensional, and W that is m-dimensional. The tensor product of these two vector spaces is nm-dimensional. Here is how it works.

Let $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ be the basis system of V, and similarly $\{\vec{f}_1, \vec{f}_2, \cdots, \vec{f}_m\}$ that of W. We now define nm basis vectors $\vec{e}_i \otimes \vec{f}_j$, where $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Never mind what $\otimes$ means at this moment. We will be more explicit later. Just regard it a symbol that defines a new set of basis vectors. This is why the word "tensor" is used for this: the basis vectors have two indices.

3.2 Vectors

We use the same notation for the column vectors as in Section 2.2.

The tensor product is *bilinear*, namely linear in V and also linear in W. (If there are more than two vector spaces, it is *multilinear*.) What it implies is that $\vec{v} \otimes \vec{w} = (\sum_i^n v_i \vec{e}_i) \otimes (\sum_j^m w_j \vec{f}_j) = \sum_i^n \sum_j^m v_i w_j (\vec{e}_i \otimes \vec{f}_j)$. In other words, it is a vector in $V \otimes W$, spanned by the basis vectors $\vec{e}_i \otimes \vec{f}_j$, and the coefficients are given by $v_i w_j$ for each basis vector.

One way to make it very explicit is to literally write it out. For example, let us take $n = 2$ and $m = 3$. The tensor product space is $nm = 6$ dimensional. The new basis vectors are $\vec{e}_1 \otimes \vec{f}_1$, $\vec{e}_1 \otimes \vec{f}_2$, $\vec{e}_1 \otimes \vec{f}_3$, $\vec{e}_2 \otimes \vec{f}_1$, $\vec{e}_2 \otimes \vec{f}_2$, and $\vec{e}_2 \otimes \vec{f}_3$. Let us write them as a six-component column vector,

$$\vec{e}_1 \otimes \vec{f}_1 = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right), \quad \vec{e}_1 \otimes \vec{f}_2 = \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right), \quad \vec{e}_1 \otimes \vec{f}_3 = \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right),$$

$$\vec{e}_2 \otimes \vec{f}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 \otimes \vec{f}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 \otimes \vec{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{pmatrix}. \tag{16}$$

Here, I drew a horizontal lines to separate two sets of basis vectors, the first set that is made of $\vec{e}_1$, and the second made of $\vec{e}_2$. For general vectors $\vec{v}$ and $\vec{w}$, the tensor product is

$$\vec{v} \otimes \vec{w} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3 \end{pmatrix}. \tag{17}$$

3.3 Matrices

A matrix is mathematically a linear map from a vector space to another vector space. Here, we specialize to the maps from a vector space to the same one because of our interest in applications to quantum mechanics, $A : V \to V$, *e.g.*,

$$\vec{v} \longmapsto A\vec{v}, \tag{18}$$

or

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_V \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_V. \tag{19}$$

On the tensor product space, the same matrix can still act on the vectors, so that $\vec{v} \mapsto A\vec{v}$, but $\vec{w} \mapsto \vec{w}$ untouched. This matrix is written as $A \otimes I$, where I is the identity matrix. In the previous example of $n = 2$ and $m = 3$, the matrix A is two-by-two, while $A \otimes I$ is six-by-six,

$$A \otimes I = \left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array} \right). \tag{20}$$

The reason why this expression is appropriate is clear once you act it on $\vec{v} \otimes \vec{w}$,

$$\begin{aligned} (A \otimes I)(\vec{v} \otimes \vec{w}) &= \left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array} \right) \left(\begin{array}{c} v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3 \end{array} \right) \\ &= \left(\begin{array}{c} (a_{11} v_1 + a_{12} v_2) w_1 \\ (a_{11} v_1 + a_{12} v_2) w_2 \\ (a_{11} v_1 + a_{12} v_2) w_3 \\ \hline (a_{21} v_1 + a_{22} v_2) w_1 \\ (a_{21} v_1 + a_{22} v_2) w_2 \\ (a_{21} v_1 + a_{22} v_2) w_3 \end{array} \right) = (A\vec{v}) \otimes \vec{w}. \end{aligned} \tag{21}$$

Clearly the matrix A acts only on $\vec{v} \in V$ and leaves $\vec{w} \in W$ untouched.

Similarly, the matrix $B : W \to W$ maps $\vec{w} \mapsto B\vec{w}$. It can also act on $V \otimes W$ as $I \otimes B$, where

$$I \otimes B = \left(\begin{array}{ccc|ccc} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{array} \right). \tag{22}$$

It acts on $\vec{v} \otimes \vec{w}$ as

$$
\begin{aligned}
(I \otimes B)(\vec{v} \otimes \vec{w}) &= \left(\begin{array}{ccc|ccc}
b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\
b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\
b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\
0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\
0 & 0 & 0 & b_{31} & b_{32} & b_{33}
\end{array}\right)
\left(\begin{array}{c}
v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3
\end{array}\right) \\
&= \left(\begin{array}{c}
v_1(b_{11}w_1 + b_{12}w_2 + b_{13}w_3) \\
v_1(b_{21}w_1 + b_{22}w_2 + b_{23}w_3) \\
v_1(b_{31}w_1 + b_{32}w_2 + b_{33}w_3) \\
\hline
v_2(b_{11}w_1 + b_{12}w_2 + b_{13}w_3) \\
v_2(b_{21}w_1 + b_{22}w_2 + b_{23}w_3) \\
v_2(b_{31}w_1 + b_{32}w_2 + b_{33}w_3)
\end{array}\right) = \vec{v} \otimes (B\vec{w}). \quad (23)
\end{aligned}
$$

In general, $(A \otimes I)(\vec{v} \otimes \vec{w}) = (A\vec{v}) \otimes \vec{w}$, and $(I \otimes B)(\vec{v} \otimes \vec{w}) = \vec{v} \otimes (B\vec{w})$.

If you have two matrices, their multiplications are done on each vector space separately,

$$
\begin{aligned}
&(A_1 \otimes I)(A_2 \otimes I) = (A_1 A_2) \otimes I, \\
&(I \otimes B_1)(I \otimes B_2) = I \otimes (B_1 B_2), \\
&(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I) = (A \otimes B). \qquad (24)
\end{aligned}
$$

The last expression allows us to write out $A \otimes B$ explicitly,

$$
A \otimes B = \left(\begin{array}{ccc|ccc}
a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} \\
a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\
a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} \\
\hline
a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} \\
a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\
a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33}
\end{array}\right). \qquad (25)
$$

It is easy to verify that $(A \otimes B)(\vec{v} \otimes \vec{w}) = (A\vec{v}) \otimes (B\vec{w})$.

Note that not every matrix on $V \otimes W$ can be written as a tensor product of a matrix on V and another on W.

Other useful formulae are

$$
\det(A \otimes B) = (\det A)^m (\det B)^n, \qquad (26)
$$

$$
\mathrm{Tr}(A \otimes B) = (\mathrm{Tr} A)(\mathrm{Tr} B). \qquad (27)
$$

In *Mathematica*, you can define the tensor product of two square matrices by

```
Needs["LinearAlgebra`MatrixManipulation`"];
KroneckerProduct[a_?SquareMatrixQ, b_?SquareMatrixQ] :=
  BlockMatrix[Outer[Times, a, b]]
```

and then

```
KroneckerProduct[A,B]
```

3.4 Tensor Product in Quantum Mechanics

In quantum mechanics, we associate a Hilbert space for each dynamical degree of freedom. For example, a free particle in three dimensions has three dynamical degrees of freedom, p_x, p_y, and p_z. Note that we can specify only p_x or x, but not both, and hence each dimension gives only one degree of freedom, unlike in classical mechanics where you have two. Therefore, momentum eigenkets $P_x|p_x\rangle = p_x|p_x\rangle$ from a basis system for the Hilbert space $\mathcal{H}_x$, etc.

The eigenstate of the full Hamiltonian is obtained by the tensor product of momentum eigenstates in each direction,

$$|p_x, p_y, p_z\rangle = |p_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle. \tag{28}$$

We normally don't bother writing it this way, but it is nothing but a tensor product.

The operator P_x is then $P_x \otimes I \otimes I$, and acts as

$$(P_x \otimes I \otimes I)(|p_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle) = (P_x|p_x\rangle) \otimes (I|p_y\rangle) \otimes (I|p_z\rangle) = p_x|p_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle \tag{29}$$

and similarly for P_y and P_z.

The Hamiltonian is actually

$$H = \frac{1}{2m}\left((P_x \otimes I \otimes I)^2 + (I \otimes P_y \otimes I)^2 + (I \otimes I \otimes P_z)^2\right). \tag{30}$$

Clearly we don't want to write the tensor products explicitly each time!

3.5 Addition of Angular Momenta

Using the example of $n = 2$ and $m = 3$, one can discuss the addition of $s = 1/2$ and $l = 1$. V is two-dimensional, representing the spin 1/2, while W is three-dimensional, representing the orbital angular momentum one. As usual, the basis vectors are

$$|\frac{1}{2},\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{31}$$

for $s = 1/2$, and

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{32}$$

for $l = 1$. The matrix representations of the angular momentum operators are

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{33}$$

while

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{34}$$

On the tensor product space $V \otimes W$, they become

$$S_+ \otimes I = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), I \otimes L_+ = \hbar \left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$S_- \otimes I = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right), I \otimes L_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{array}\right),$$

$$S_z \otimes I = \frac{\hbar}{2}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right), I \otimes L_z = \hbar \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right). \tag{35}$$

The addition of angular momenta is done simply by adding these matrices, $J_{\pm,z} = S_{\pm,z} \otimes I + I \otimes L_{\pm,z}$,

$$J_+ = \hbar \left(\begin{array}{ccc|ccc} 0 & \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$J_- = \hbar \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{array}\right),$$

$$J_z = \hbar \left(\begin{array}{ccc|ccc} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right). \tag{36}$$

It is clear that

$$|\frac{3}{2}, \frac{3}{2}\rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right), \qquad |\frac{3}{2}, -\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right). \tag{37}$$

They satisfy $J_+|\frac{3}{2},\frac{3}{2}\rangle = J_-|\frac{3}{2},-\frac{3}{2}\rangle = 0$. By using the general formula $J_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1\rangle$, we find

$$J_-|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ \sqrt{2} \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle, \qquad J_+|\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ \sqrt{2} \\ 0 \end{array}\right) = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle. \tag{38}$$

We normally write them without using column vectors,

$$\begin{aligned} J_-|\frac{3}{2},\frac{3}{2}\rangle &= J_-|\frac{1}{2},\frac{1}{2}\rangle \otimes |1,1\rangle \\ &= |\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,1\rangle + \sqrt{2}|\frac{1}{2},\frac{1}{2}\rangle \otimes |1,0\rangle = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle, \qquad (39) \\ J_+|\frac{3}{2},-\frac{3}{2}\rangle &= J_+|\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,-1\rangle \\ &= |\frac{1}{2},\frac{1}{2}\rangle \otimes |1,-1\rangle + \sqrt{2}|\frac{1}{2},-\frac{1}{2}\rangle \otimes |1,0\rangle = \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle. \qquad (40) \end{aligned}$$

Their orthogonal linear combinations belong to the $j = 1/2$ representation,

$$|\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \qquad |\frac{1}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{41}$$

It is easy to verify $J_+|\frac{1}{2},\frac{1}{2}\rangle = J_-|\frac{1}{2},-\frac{1}{2}\rangle = 0$, and $J_-|\frac{1}{2},\frac{1}{2}\rangle = |\frac{1}{2},-\frac{1}{2}\rangle$.

Now we can go to a new basis system by a unitary rotation

$$U = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 \end{array}\right). \tag{42}$$

This matrix puts transpose of vectors in successive rows so that the vectors of definite j and m are mapped to simply each component of the vector,

$$
\begin{gathered}
U|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right), U|\frac{3}{2},\frac{1}{2}\rangle = \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right), U|\frac{3}{2},-\frac{1}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right), U|\frac{3}{2},-\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right), \\
U|\frac{1}{2},\frac{1}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \end{array}\right), U|\frac{3}{2},\frac{3}{2}\rangle = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \end{array}\right).
\end{gathered}
\tag{43}
$$

For the generators, we find

$$
UJ_zU^\dagger = \hbar \left(\begin{array}{cccc|cc}
\frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3}{2} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}
\end{array}\right), \tag{44}
$$

$$
UJ_+U^\dagger = \hbar \left(\begin{array}{cccc|cc}
0 & \sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right), \tag{45}
$$

$$
UJ_-U^\dagger = \hbar \left(\begin{array}{cccc|cc}
0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right). \tag{46}
$$

Here, the matrices are written in a block-diagonal form. What we see here is that this space is actually a direct sum of two representations $j = 3/2$ and

$j = 1/2$. In other words,

$$V_{1/2} \otimes V_1 = V_{3/2} \oplus V_{1/2}. \tag{47}$$

This is what we do with Clebsch–Gordan coefficients. Namely that the Clebsch–Gordan coefficients define a unitary transformation that allow us to *decompose* the tensor product space into a direct sum of *irreducible* representations of the angular momentum. As a mnemonic, we often write it as

$$2 \otimes 3 = 4 \oplus 2. \tag{48}$$

Here, the numbers refer to the dimensionality of each representation $2j + 1$. If I replace $\otimes$ by $\times$ and $\oplus$ by $+$, my eight-year old daughter can confirm that this equation is true. The added meaning here is that the right-hand side shows a specific way how the tensor product space decomposes to a direct sum.