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Renormalizing the SM Lagrangian for Precision Tests

1. SM Lagrangian $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$\mathcal{L} = \mathcal{L}\Big(\underbrace{g_3,\ g_2,\ g_1}_{\text{gauge interactions}},\ \underbrace{\lambda,\ v}_{\text{Higgs sector}},\ m_f \Big)$$

Yukawa interactions

Beyond tree level, we need to renormalize couplings (those listed above) and fields (wavefunction renormalization)

2. $SU(3)_c$ coupling α_s

(1) $\alpha_s \equiv \dfrac{g_3^2}{4\pi}$

is usually defined in the $\overline{MS}$-scheme.

⇒ Continuing momentum integrals from

4 to $n \equiv 4-2\varepsilon$ dimensions,

and then

subtracting off $\left(\dfrac{1}{\varepsilon} - \gamma_E + \ln 4\pi\right)$.

<u>Note</u>: To preserve the dimensionless nature of the coupling,

$$g_3 \longrightarrow g_3 \cdot \mu^{\varepsilon} \quad \left(\text{in } n = 4-2\varepsilon \text{ dim.}\right)$$

⇒ A factor $\ln \mu^2$ always comes with $\dfrac{1}{\varepsilon}$.

⇒ Effective QCD coupling $\alpha_s(\mu)$ with

$$\mu \frac{\partial \alpha_s}{\partial \mu} = 2\beta(\alpha_s) = -\frac{\beta_0}{2\pi}\alpha_s^2 - \frac{\beta_1}{4\pi^2}\alpha_s^3 - \cdots \quad (\beta_2, \beta_3)$$

$$\beta_0 = 11 - \frac{2}{3} n_f$$

n_f : # of quarks with mass less than μ

(2) The new convention is to choose $\mu_0 = M_Z$
$= 91.1876 \pm 0.0021$ GeV

and calculate

$\alpha_s(\mu)$

from
$$\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} = \log\left(\frac{\mu^2}{\mu_0^2}\right)$$

with
$$\alpha_s(M_Z) = 0.118 \pm 0.003$$

(3) One can also introduce $\Lambda_{QCD}\ (\equiv \Lambda)$ to parametrize the μ dependence of $\alpha_s(\mu)$.

The definition of Λ is arbitrary. One way is to define it via

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \Big[1 - \underset{(\beta_1, \beta_2)}{\cdots} \Big]$$

Note: $\alpha_s(\mu) \to 0$ as $\mu \to \infty$.

$$\Lambda_{QCD}^{(5)} = 380 \pm 60 \text{ MeV} \quad \left(\text{with top quark decoupled}\right)$$

Note. β_0 and β_1 are independent of renormalization scheme.

3. We can trade (g_2, g_1, v)

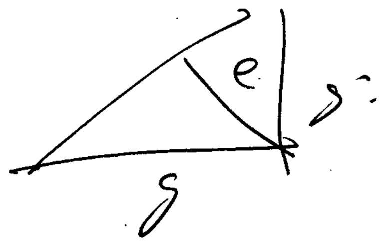
with (α_{em}, M_W, M_Z)

(1) The tree level relations, denoted by subscript (0),

$$\alpha_{em}^{(0)} = \frac{1}{4\pi} \frac{g_2^2 g_1^2}{g_2^2 + g_1^2}$$

$$M_W^{(0)} = \frac{1}{2} g_2 v$$

$$M_Z^{(0)} = \frac{1}{2} \sqrt{g_2^2 + g_1^2}\, v$$



(2) Conventionly, we use on-shell definition and define

$\alpha_{em}(0) \equiv \alpha(0)$, from $(g-2)_e$, etc.

$M_W \equiv$ on-shell mass (pole mass)

$M_Z \equiv$ on-shell mass (pole mass)

(3)

Parameter	Measured Value	Precision
$\alpha_{em}^{(0)}$	$[137.03599911(46)]^{-1}$	$4. \times 10^{-9}$
M_Z	91.1876 (21)	2×10^{-5}
M_W	80.454 (59)	74×10^{-5}

Note. The precision in M_W measurement is poor. Thus, we trade M_W with G_F as input data to fix SM parameters

G_F	$1.16637(1) \times 10^{-5}$ GeV^{-2}	1×10^{-5}

The Fermi constant G_F is determined from muon lifetime

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1+\frac{3}{5}\frac{m_\mu^2}{M_W^2}\right)\left[F\left(\frac{m_e^2}{m_\mu^2}\right)\right] \cdot \left[1+\left(\frac{25}{8}-\frac{\pi^2}{2}\right)\frac{\alpha}{\pi}\left(1+\frac{2\alpha}{3\pi}\ln\left(\frac{m_\mu}{m_e}\right)\right)\right].$$

with $\quad F(x) = 1-8x+8x^3-x^4-12x^2\ln x$

4. (α_{em}, G_F, M_Z) scheme

This is the (α_s, M_W, M_Z) on-shell scheme with G_F (instead of M_W) as one of the input data. (At tree level, $v^2 = \frac{1}{\sqrt{2}\, G_F} = (246\,\mathrm{GeV})^2$)

(1) Renormalization :

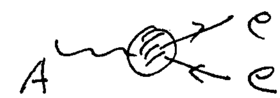
① replace

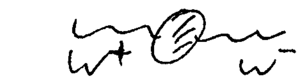
$$e \longrightarrow e_0 = e + \delta e$$

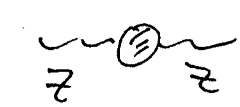
$$M_{W,Z}^2 \longrightarrow M^{0\,2}_{W,Z} = M^2_{W,Z} + \delta M^2_{W,Z}$$

bare measurable counterterm

② fix counterterms by conditions, e.g.

$\delta e = -$  at $q^2 = 0$, $q^0 \to 0$

$\delta M_W^2 =$  $\Big|_{q^2 = M_W^2}$

$\delta M_Z^2 =$  $\Big|_{q^2 = M_Z^2}$

(2) α_{em}

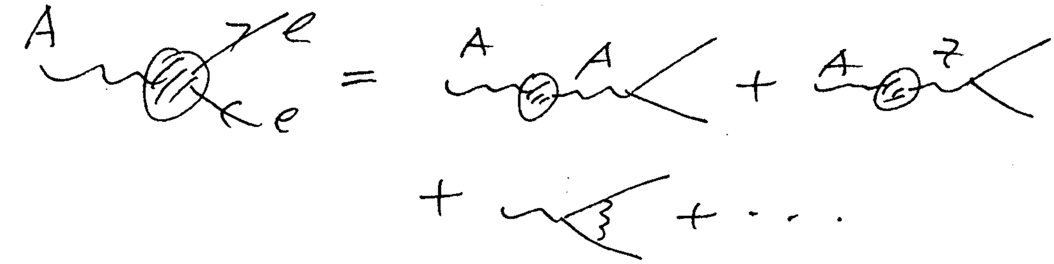
1 Denote the vacuum polarization functions as (2-point functions)

$$\Pi^{ij}_{\mu\nu}(q^2) \equiv -i g_{\mu\nu}\left[A^{ij}(0) + q^2 F^{ij}(q^2)\right] + \begin{pmatrix} q_\mu q_\nu \\ \text{terms} \end{pmatrix}$$

$i, j = A, W^\pm, Z$

$$\equiv -i g_{\mu\nu}\left(\Pi_{ij}\right) + \begin{pmatrix} q_\mu q_\nu \\ \text{terms} \end{pmatrix}$$

(don't contribute for gauge bosons coupling to massless fermions)

2 

$$\frac{\delta\alpha}{\alpha} = 2\,\frac{\delta e}{e} = F^{AA}(0) + 2\,\frac{s_W}{c_W}\,\frac{A^{AZ}(0)}{M_Z^2}$$

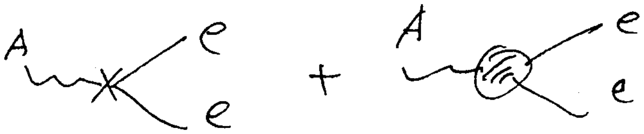
where $s_W^2 \equiv 1 - \frac{M_W^2}{M_Z^2}$ (definition of $s_W \equiv \sin\theta_W$)

and $c_W = \frac{M_W}{M_Z}$

<u>Note</u>: For photon, $\Pi_{AA} = q^2 F^{AA}(q^2)$,

i.e. $A^{AA}(0) = 0$. ($U(1)_{em}$ gauge invariance. Otherwise, photon will gain mass.)

③ When calculating observables at Z-pole, $q^2 = M_Z^2$, one usually encounters



$$\alpha(0)\left\{\left(1+\frac{\delta\alpha}{\alpha}\right)(\cdots) + (-1)\, F^{AA}(M_Z^2) + \cdots \right\}$$

⇒ Rewrite

Cancelled

$$2\frac{\delta e}{e} = F^{AA}(0) + \cdots$$

$$= \underbrace{F^{AA}(0) - F^{AA}(M_Z^2)}_{\Delta\alpha,\ \text{finite (but large)}} + F^{AA}(M_Z^2) + \cdots$$

$$\sim Q_f^2 \ln\frac{M_Z^2}{m_f^2} + \cdots$$

⇒ Instead of writing the result as

$$\alpha(0)\left(1+\Delta\alpha + \cdots\right)$$

we can write

$$\alpha(M_Z^2)\left(1+\cdots\right),$$

i.e. the large correction $\Delta\alpha$ is absorbed into $\alpha(M_Z^2)$

⇒ Resummation

$$\alpha(M_Z^2) = \frac{\alpha(0)}{1-\Delta\alpha}$$

④ what's $\Delta\alpha$?

$$\Delta\alpha \equiv F^{AA}(0) - F^{AA}(M_Z^2)$$

with only light fermions ($m_f < M_Z$) included.

Namely, we have taken $\frac{M_Z^2}{m_t^2} \to 0$ limit.

(decoupled)

$$\Delta\alpha = \sum_{\text{leptons}}$$  $$+ \ (\Delta\alpha)^{(5)}_{had}$$

(for 5 light quark flavors)

from $e^-e^+ \to$ hadrons data

$$(\Delta\alpha)_{leptons} = \frac{\alpha}{3\pi}\sum_{\ell}\left[\ln\left(\frac{M_Z^2}{m_\ell^2}\right) - \frac{5}{3} + \frac{\alpha}{\pi}(\cdots)\right]$$

$$= 0.0314966 \pm 0.\underbrace{000000}_{6}4$$

$$(\Delta\alpha)^{(5)}_{had} = \frac{-M_Z^2}{4\pi^2\alpha}\,\mathrm{Re}\int_{4m_\pi^2}^{\infty} ds \,\frac{\sigma(e^-e^+ \to \text{hadrons})}{s - M_Z^2 - i\varepsilon}$$

$$= 0.02766 \pm 0.00013$$

$\Rightarrow$ The uncertainty in $(1+\Delta\alpha)$ is dominated by the error in $(\Delta\alpha)^{(5)}_{had}$, which yields about $\frac{0.00013}{1.0592} = 0.11\,\%$ accuracy.

(3) M_Z:

(1) The dressed propagator:

$$Z \sim\sim\sim Z + Z \sim\sim\bigcirc\sim\sim Z + \cdots$$

$$= \frac{-i g_{\mu\nu}}{q^2 - M_Z^{0\,2} + A^{ZZ}(0) + q^2 F^{ZZ}(q^2)} + (q_\mu q_\nu \text{ term})$$

(2)

$$M_Z^{0\,2} = M_Z^2 + \delta M_Z^2$$

Expand $q^2 F^{ZZ}(q^2)$ around $q^2 = M_Z^2$,

$$q^2 F^{ZZ}(q^2) = M_Z^2 F^{ZZ}(M_Z^2) + \left\{ F^{ZZ}(M_Z^2) + M_Z^2 \left.\frac{d F^{ZZ}(q^2)}{d q^2}\right|_{q^2 = M_Z^2} \right\} \cdot (q^2 - M_Z^2) + \cdots$$

Using on-shell subtraction scheme,

$$\delta M_Z^2 = A^{ZZ}(0) + M_Z^2 F^{ZZ}(M_Z^2)$$

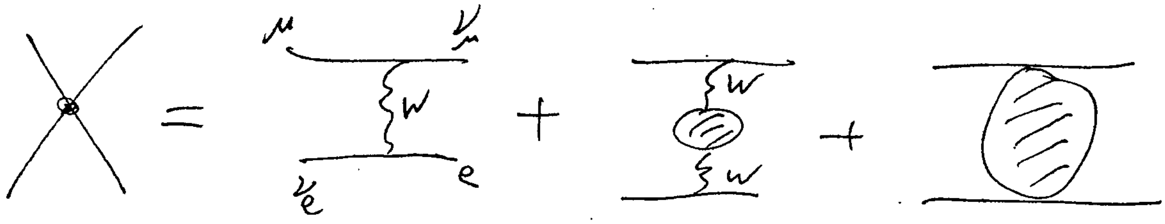
$$\Rightarrow \quad \frac{-i g_{\mu\nu}}{q^2 - M_Z^2} \underbrace{\left\{ 1 - F^{ZZ}(M_Z^2) - M_Z^2 \frac{d F^{ZZ}(M_Z^2)}{d q^2} \right\}}$$

Wavefunction renormalization $(1 - \delta Z_Z)$

$\left(\text{for } Z^0 \to Z_Z^{1/2} Z = (1 + \tfrac{1}{2}\delta Z_Z) Z \right)$,

(4) M_W :

Given (α_{em}, G_F, M_Z) and (m_t, m_H), one can predict M_W by correlating G_F to τ_μ (μ-lifetime)



$$G_F = \frac{\pi \alpha(0)}{\sqrt{2}} \frac{1}{M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right)} (1+\Delta r)$$

$$\Delta r = \frac{A^{WW}(0)}{M_W^2} + \cdots$$

$$= \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho + (\Delta r)_{rem}(m_t, m_H)$$

6% 3% 1%

where
$$\Delta\rho \equiv \frac{A_{ZZ}(0)}{M_Z^2} - \frac{A_{WW}(0)}{M_W^2}$$

$$= \frac{3 G_F m_t^2}{8\pi^2\sqrt{2}} + \cdots$$

$$= 0.00992 \left(\frac{m_t}{177.9\,\mathrm{GeV}}\right)^2 + \cdots$$

uncertainty from $\alpha(M_Z)$

$$\Rightarrow \Delta r = 0.03434 \pm 0.0017 \pm 0.00014$$

Resum $\Delta\alpha$, $\Delta\rho$

$$(1+\Delta r) \longrightarrow \frac{1}{1-\Delta\alpha}\,\frac{1}{1+\frac{c_W^2}{s_W}\Delta\rho} + (\Delta r)_{rem}$$

$$\longrightarrow \frac{1}{1-\Delta\alpha}\,\frac{1}{1+\frac{c_W^2}{s_W^2}\Delta\rho - (\Delta r)_{rem}}$$

$$\Rightarrow \quad G_F = \frac{\pi\alpha}{\sqrt{2}}\,\frac{1}{M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right)}\,\frac{1}{1-\Delta r}$$

with $\quad (1-\Delta r) = (1-\Delta\alpha)\left(1+\frac{c_W^2}{s_W^2}\Delta\rho\right) - \Delta r_{rem}$

Therefore, M_W can be predicted from solving

$$\left(1-\frac{M_W^2}{M_Z^2}\right)\frac{M_W^2}{M_Z^2} = \frac{\pi\alpha(0)}{\sqrt{2}\,M_Z^2 G_F\,(1-\Delta r)}$$

$\Rightarrow$ From Z-pole data and m_t measurement, SM predicts

$$M_W = 80.378 \pm 0.023 \ \text{GeV}$$

Note. In the above derivation, we have identified

$$\Delta\rho \equiv \frac{A_{ZZ}(0)}{M_Z^2} - \frac{A_{WW}(0)}{M_W^2}$$

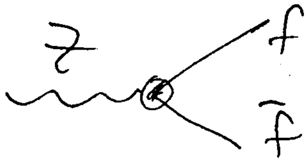
which is the dominant contribution in the ratio

$$R_\nu \equiv \frac{\sigma_{\nu N}^{NC}}{\sigma_{\nu N}^{CC}}$$

with

$$\frac{G_{NC}(0)}{G_{CC}(0)} \equiv \frac{1}{1-\Delta\rho}, \qquad \left(G_{CC}(0) \equiv G_F \right)$$

4. Predict Z-boson observables from SM



g_V^f, g_A^f and $(G_F M_Z^2)$
normalization

(1)

$$\Gamma_{Z\to f\bar{f}} \sim (g_V^f)^2 + (g_A^f)^2 \longrightarrow \Gamma_{tot},\ R_{had},\ \sigma_{peak}$$

$$\left.\begin{aligned} A_{FB}^f &= \frac{3}{4}\,\frac{2 g_V^e g_A^e}{g_V^{e2} + g_A^{e2}} \cdot \frac{2 g_V^f g_A^f}{g_V^{f2} + g_A^{f2}} \\ P_\tau &= \frac{2\, g_V^\tau g_A^\tau}{g_V^{\tau 2} + g_A^{\tau 2}} \\ A_{LR} &= \frac{2 g_V^e g_A^e}{g_V^{e2} + g_A^{e2}} \end{aligned}\right\} \text{depend on ratio}$$

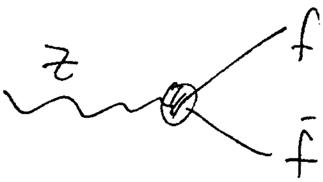
$$\frac{g_V^f}{g_A^f} \longleftrightarrow \sin^2\theta_f$$

$$\Rightarrow \frac{g_V^f}{g_A^f} \text{ provides precision tests of SM}$$

(2) Effective Couplings from SM. (for $f \neq b$)

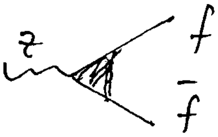
$$g_A^f = I_3^f \sqrt{\rho_f}$$

$$g_V^f = \left(I_3^f - 2Q_f s_f^2\right)\sqrt{\rho_f}$$

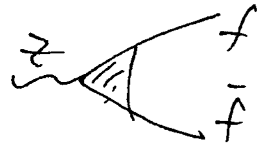


where $s_f^2 \equiv \sin^2\theta_f$

and

$$\rho_f = 1 + \left(m_t, \Delta r\right) + $$ 

$$= \underbrace{1 + \Delta\rho} + \cdots$$

$$s_f^2 = \underbrace{s_W^2\left(1 + \frac{c_W^2}{s_W^2}\Delta\rho\right)}_{\text{universal}} + \cdots + $$ 

f-dependent
(m_t, m_H independent except $f = b$)

Usually,

$$s_f^2 \equiv (1 + \Delta\kappa)\, s_W^2 \equiv \kappa\, s_W^2, \qquad \left(s_W^2 \equiv 1 - \frac{M_W^2}{M_Z^2}\right)$$

so that

$$\Delta\kappa = \frac{c_W^2}{s_W^2}\Delta\rho + \cdots$$

Note:

$$\sin^2\theta_\ell = 0.2322 \pm 0.0004 \qquad \text{(for } f = \text{lepton)}$$

$$\sin^2\theta_W = 0.2249 \pm 0.0013$$

(3) The partial decay width

$$\Gamma(Z \to f\bar{f}) = \frac{N_c G_F M_Z^3}{24\pi\sqrt{2}} \rho_f \left[\left(1 - 4|Q_f| s_f^2\right)^2 + 1 \right]$$

with

$$s_f^2 \equiv (1+\Delta k) s_W^2 \equiv k\, s_W^2$$

$$\Rightarrow \rho_f = \frac{1-\Delta r}{1+\Delta Z} + \cdots$$

$$\Sigma_Z(q^2) = \Pi_{ZZ}(q^2) - \frac{\left(\Pi_{ZA}(q^2)\right)^2}{q^2 + \Pi_{AA}(q^2)}$$

$$\Delta Z = \mathrm{Re}\, \Sigma_Z'(M_Z^2)$$

$$= -1 + (1-\Delta\alpha)(1-\Delta\rho)\left(1+\frac{c^2}{s^2}\Delta\rho\right) + \Delta Z_{rem}$$

$$\Rightarrow \rho_f = \frac{1}{1-\Delta\rho} - \frac{\Delta r_{rem} + \Delta Z_{rem}}{1-\Delta\alpha} + \cdots$$

Also

$$k = 1 + \frac{c^2}{s^2}\Delta\rho - \frac{c}{s}\frac{\Pi_{ZA}(M_Z^2)}{M_Z^2 + \Pi_{AA}(M_Z^2)} + \cdots$$

$$= 1 + \frac{c^2}{s^2}\Delta\rho - \frac{c}{s}\frac{1}{1-\Delta\alpha}\left(\frac{\Pi_{ZA}(M_Z^2)}{M_Z^2}\right)_{rem} + \cdots$$

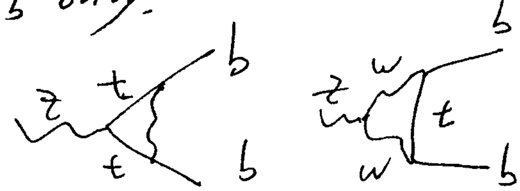
(4) For $Z \to b\bar{b}$

$$\rho_f \longrightarrow \rho_b = \rho_f (1+\tau)^2 \quad ;$$

$$\kappa \longrightarrow \kappa_b = \frac{\kappa}{1+\tau}$$

with
$$\tau = -2\left(\frac{G_F m_t^2}{8\pi^2\sqrt{2}}\right) = -2\left(\frac{m_t}{4\pi v}\right)^2, \quad \left(v^2 = \frac{1}{\sqrt{2}\,G_F}\right)$$

new contribution to $Z \to b\bar{b}$ only.

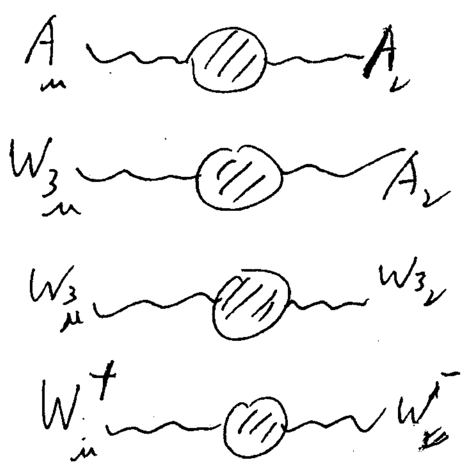


5. New physics and Precision data

(1) In the SM, the physical observables $(M_W,\ g_V^f,\ g_A^f,\ \ldots)$, or $(\Delta r,\ \rho_f,\ \kappa,\ \ldots)$ are dominated by $\Delta\alpha$, $\Delta\rho$, which are defined by the vacuum polarization functions of A_μ, Z_μ, W_μ^+, W_μ^-.

(2) Assume new physics effect only comes in via those 2-point functions (Oblique corrections).

$\Rightarrow$ There are 4 of them



$\Pi_{QQ}(q^2)$

$\Pi_{3Q}(q^2)$

$\Pi_{33}(q^2)$

$\Pi_{WW}(q^2)$

(3) If the New physics scale Λ_{new} is much larger than M_Z, i.e. $\Lambda_{new} \gg M_Z$, then such effects can be described by 3 parameters (S, T, U) at the 1-loop level.

$$\alpha T = \Delta\rho = \frac{A^{33}(0)}{M_Z^2} - \frac{A^{WW}(0)}{M_W^2}$$

$$\frac{\alpha S}{4 s_W^2} = \frac{1}{s_W} F^{3Q}(M_Z^2) - F^{33}(M_Z^2)$$

$$\frac{\alpha U}{4 s_W^2} = F^{WW}(M_W^2) - F^{33}(M_Z^2)$$

Note: $\Delta\alpha = F^{QQ}(0) - F^{QQ}(M_Z^2)$

$$\Pi^{ij}_{\mu\nu}(q^2) \equiv -i g_{\mu\nu}\left[A^{ij}(0) + q^2 F^{ij}(q^2)\right] + \left(\begin{matrix} q_\mu q_\nu \\ \text{terms} \end{matrix}\right)$$

assumed not to contribute

$\Delta\alpha$	$\Delta\rho = \alpha T$	U	S
heavy Object decoupled light: $\sim Q_f^2 \ln \frac{M_Z^2}{m_f^2}$	I-breaking (m_t, m_b) ——— dominated by top	I-breaking (small) (in Wavefunctions) ——— ~ 0 in technicolor	I-Conserving ——— Sensitive to SM Higgs ——— heavy degenerate f-multiplet heavy quark doublet (sequential) $\simeq N_c \frac{G_F M_W^2}{12\pi^2\sqrt{2}}$ ——— technicolor $(N_{TC}=4)$ $\frac{\alpha}{4 s_w^2} \begin{pmatrix} 0.04 \leftarrow \text{1-doublet} \\ 2.1 \leftarrow \text{(1-generation)} \end{pmatrix}$

<u>Note</u>. M_H Contributions in SM:

m_t Contribution

$$\alpha \cdot \Delta T = \frac{-3}{16\pi C_W^2} \ln \frac{m_H^2}{M_Z^2} + 3\left(\frac{m_t}{4\pi v}\right)^2$$

$$\alpha \cdot \Delta S = \frac{1}{12\pi} \ln \frac{m_H^2}{M_Z^2}$$

$$\alpha \cdot \Delta U \simeq 0$$

(5) In the SM, relations between $(\Delta r, \rho_f, \kappa)$ and (S, T, U)

$$\alpha S \simeq 4s^2\left(c^2(\Delta r_W + \Delta\rho) + (c^2 - s^2)\Delta\kappa\right) \longrightarrow 0.(\Delta\rho)$$

$$\alpha T \simeq \Delta\rho$$

$$\alpha U \simeq 4s^2\left(s^2(\Delta r_W + 2\Delta\kappa) - c^2\Delta\rho\right) \longrightarrow 0.(\Delta\rho)$$

where $\rho_f \equiv 1 + \Delta\rho$

$\kappa \equiv 1 + \Delta\kappa$, with $\Delta\kappa \simeq \frac{c^2}{s^2}\Delta\rho + \cdots$

$(1 - \Delta r_{oblique}) = (1 - \Delta\alpha)(1 - \Delta r_W)$

with $\Delta r_W \simeq \frac{-c^2}{s^2}\Delta\rho + \cdots$

$$\Delta\rho \simeq 3\left(\frac{m_t}{4\pi v}\right)^2 + \cdots$$